Recognizing the Cartan association schemes in polynomial time

(based on the joint work with A.Vasil'ev)

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The problem statement

Definition.

For a permutation group $G \leq \mathrm{Sym}(\Omega)$, the colored graph Γ_G is defined to be the compete graph with vertex set Ω and the color classes $\mathrm{Orb}(G, \Omega \times \Omega)$.

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When $|G|$ is odd, the problem CAUT can be solved in time $n^{O(1)}$ with $n = |\Omega|$ (P, 2012).

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- The *m*-dim intersection numbers are, roughly speaking, the intersection numbers for G acting on Ω^m.
- If G is transitive and H is the point stabilizer, then Ω can be identified with G/H so that $(Hx)^g = Hxg$ for all $x \in G$.

Groups with BN-pairs

In any group G with a BN-pair we have a pair of subgroups B and N such that:

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The main result

Notation.

Denote by $\mathrm{Car}(m, q)$ the class of all simple $G \leq \mathrm{Sym}(\Omega)$ s.t.

- G is a group of Lie type of rank m over a field of order q,
- $\Omega = G/H$, where H is a Cartan subgroup,
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Theorem 1.

There are constants c_m, c_q s.t. if $m \geq c_m$, $q \geq c_q m$, then CAUT can be solved in time $n^{O(1)}$ for any $G \in \mathrm{Car}(m, q)$ $(n = |\Omega|)$.

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Theorem 2.

Under the hypothesis of Theorem 1, suppose $G \in \mathrm{Car}(m, q)$. Then the association scheme of G is uniquely determined by the 2-dim intersection numbers.

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Remarks:

- Step 1 is performed by the Weisfeiler-Leman algorithm.
- $|G| \leq n^2$, where $n = |\Omega|$.
- The running time is $n^{O(1)}$.

The correctness of the algorithm

Notation.

Denote by k and c the maximum color valency of a vertex and of a pair of vertices in Γ, respectively.

Theorem 3.

Suppose that the algorithm finds the group G. Then $G = \mathrm{Aut}(\Gamma)$ whenever $2c(k-1) < n$.

Lemma.

Suppose $\Gamma = \Gamma_G$ is a Cartan graph. Then $k \leq |H|$ and

$$c \leq \max_{x \in G \setminus H} \sum_{h \in H} \chi(hx),$$

where χ is the permutation character of the group G.